PRINCIPLES OF ANALYSIS
LECTURE 7 - COUNTABILITY AND UNCOUNTABILITY

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1. Countability

Let $n \in \mathbb{N}$ and set $\mathbb{N}_n = \{0, 1, \dots, n-1\}$. By convention, we write that $|\mathbb{N}_n| = n$.

Let A be a set. We say that A is *finite* if $|A| = n$ for some $n \in \mathbb{N}$, that is, if there exists a bijective function $A \to \mathbb{N}_n$. We say that A is *infinite* if it is not finite.

We say that A is *countable* if $|A| \leq |\mathbb{N}|$, that is, if there exists an injective function $A \to \mathbb{N}$, or equivalently, if there exists a surjective function $\mathbb{N} \to A$. We say that A is *countably infinite* if it is both countable and infinite. A set is called *uncountable* if it is not countable.

Proposition 1. *Every infinite set has a countable subset.*

Proof. Let A be an infinite set. Suppose, by way of contradiction, that there does not exist an injective function from $\mathbb{N}$ to A.

For every $f : \mathbb{N} \to A$, define $F_f = \{n \in \mathbb{N} \mid f(n) = f(m) \text{ for some } m < n\}$. Since f is not injective, F_f is nonempty. Set $n_f = \min(F_f)$.

Define the set $M \subset \mathbb{N}$ by

$$M = \{n \in \mathbb{N} \mid \exists f : \mathbb{N} \to A \text{ such that } n = n_f.\}$$

Let $m = \max(M)$; then there exists $g : \mathbb{N} \to A$ such that $m = n_g$. Thus the function $g \restriction_{\mathbb{N}_m} : \mathbb{N}_m \to A$ is injective. Since A is infinite, $g \restriction_{\mathbb{N}_m}$ is not surjective, so there exists $a \in A$ such that $g(n) \neq a$ for every $n \in \mathbb{N}_m$. Define a funtion

$$h : \mathbb{N} \to A \quad \text{by} \quad h(n) = \begin{cases} g(n) & \text{if } n < m; \\ a & \text{if } n \geq m. \end{cases}$$

Now $n_h = m + 1$, so $m + 1 \in M$, contradicting $m = \max(M)$. $\square$

The Hebrew *aleph* is written $\aleph$. Cantor define $\aleph_0$ to be the cardinality of the natural numbers: $\aleph_0 = |\mathbb{N}|$. As a corollary of the previous proposition, $\aleph_0$ is the smallest of the "transfinite" cardinals.

Corollary 1. *Let A be an infinite set. Then $|\mathbb{N}| \leq |A|$.*

Proof. Let B be a countable subset of A. Then the inclusion function

$$\text{inc} : B \to A \quad \text{given by} \quad \text{inc}(b) = b$$

is injective, so $|\mathbb{N}| = |B| \leq |A|$. $\square$

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Proposition 2. *Every subset of a countable set is countable.*

Proof. Let A be a countable set and let $B \subset A$. Since A is countable, there exists an injective function $f : A \to \mathbb{N}$. Then $f \restriction_B : B \to \mathbb{N}$ is also injective, so B is countable. $\square$

Proposition 3. *Let A and B be countable sets. Then $A \cup B$ is countable.*

Proof. Since A and B are countable, there exist surjective functions $g : \mathbb{N} \to A$ and $h : \mathbb{N} \to B$. Define a function

$$f : \mathbb{N} \to A \cup B \quad \text{by} \quad f(n) = \begin{cases} g(\frac{n}{2}) & \text{if } n \text{ is even;} \\ h(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

Then f is surjective, so $A \cup B$ is countable. $\square$

Proposition 4. *Let A and B be countable sets. Then $A \times B$ is countable.*

Proof. Since A and B are countable, there exist injective functions $g : A \to \mathbb{N}$ and $h : B \to \mathbb{N}$. Define a function

$$f : A \times B \to \mathbb{N} \quad \text{by} \quad f(a,b) = 2^{g(a)} \cdot 3^{h(b)}.$$

To see that f is injective, suppose that $f(a_1, b_1) = f(a_2, b_2)$. Then $2^{g(a_1)} 3^{h(b_1)} = 2^{g(a_2)} 3^{h(b_2)}$. Thus $2^{g(a_1) - g(a_2)} = 3^{h(b_2) - h(b_1)}$, where without loss of generality $g(a_1) \geq g(a_2)$. If $g(a_1) > g(a_2)$, then 2 divides the left side and not the right; this is impossible, so $g(a_1) = g(a_2)$, and since g is injective, we must have $a_1 = a_2$. Similarly, $b_1 = b_2$. $\square$

Proposition 5. *The set $\mathbb{Z}$ of integers is a countable set.*

Proof. Define a function

$$f : \mathbb{Z} \to \mathbb{N} \quad \text{by} \quad f(n) = \begin{cases} 1 & \text{if } n = 0; \\ 2n & \text{if } n > 0; \\ 2n+1 & \text{if } n < 0. \end{cases}$$

Then f is injective, so $\mathbb{Z}$ is countable. $\square$

Proposition 6. *The set $\mathbb{Q}$ of rational numbers is a countable set.*

Proof. Let $\mathbb{Z}^+$ denote the positive integers, $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$. This is a subset of $\mathbb{Z}$, and is therefore countable. By Proposition 4, it suffices to find an injective function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$. Every rational number has a unique expression $\frac{p}{q}$ as a ratio of integers, where $\gcd(p,q) = 1$ and $q > 0$. This induces a function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$ given by $\frac{p}{q} \mapsto (p, q)$. This function is bijective; therefore $\mathbb{Q}$ is countable. $\square$

2. Cardinal Arithmetic

Let A and B be sets. We define the sum, product, and exponentiation of cardinal numbers to match that of finite numbers.

Define
$$|A| + |B| = |(A \times \{0\}) \cup (B \times \{1\})|.$$
Note that even if $A \cap B$ is nonempty, $A \times \{0\}$ and $B \times \{1\}$ are disjoint sets. So if A is any set with m elements and B is any set with n elements, then $(A \times \{0\}) \cup (B \times \{1\})$ is a set with $m + n$ elements.

Define
$$|A| \cdot |B| = |A \times B|.$$
Again, if A and B are finite with m and n elements respectively, then $A \times B$ has mn elements.

Define
$$|A|^{|B|} = |\mathcal{F}(B, A)|,$$
where $\mathcal{F}(B, A)$ denotes the set of all functions from B to A. This again agrees with the finite case.

We have seen that for any set X, there does not exist a surjective function from X to its power set $\mathcal{P}(X)$. Thus $|X| < |\mathcal{P}(X)|$. Actually, the next proposition shows that $|\mathcal{P}(X)| = 2^{|X|}$.

Proposition 7. *Let X be any set and let $T = \{0, 1\}$. Let $\mathcal{P}(X)$ denote the power set of X and let $\mathcal{F}(X, T)$ denote the set of all functions from X to T. Then $|\mathcal{P}(X)| = |\mathcal{F}(X, T)|$.*

Proof. Define a function
$$\Phi : \mathcal{F}(X, T) \to \mathcal{P}(X) \quad \text{by} \quad \Phi(f) = f^{-1}(1).$$
It suffices to show that Φ is bijective.

To see that Φ is injective, suppose that $\Phi(f_1) = \Phi(f_2)$, where $f_1 : X \to T$ and $f_2 : X \to T$. Then $f_1(x) = 1$ if and only if $f_2(x) = 1$. For $x \in X$, $f_i(x)$ is either 1 or 0, so if it is not 1, it is zero. Therefore $f_1(x) = 0$ if and only if $f_2(x) = 0$. So $f_1(x) = f_2(x)$ for every $x \in X$, that is, $f_1 = f_2$.

To see that Φ is surjective, let $A \in \mathcal{P}(X)$. Define a function
$$f : X \to T \quad \text{by} \quad f(x) = \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in A. \end{cases}$$
Then $A = f^{-1}(1)$, so $\Phi(f) = A$. $\square$

3. Intervals

We define *intervals* of real numbers as follows:

(a) $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$;
(b) $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$;
(c) $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$;
(d) $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$;
(e) $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$;
(f) $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$;
(g) $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$;
(h) $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$;
(i) $(-\infty, \infty) = \mathbb{R}$.

Intervals of types **(a)**, **(e)**, **(g)**, and **(i)** are called *open intervals*, and intervals of types **(d)**, **(f)**, **(h**, and **(i)** are called *closed intervals*.

Proposition 8. *Any two intervals have the same cardinality.*

Proof. We show part of this and leave the remaining details to the reader.

First note that the function $x \mapsto \frac{x-a}{b-a}$ maps (a, b) bijectively onto $(0, 1)$. So all intervals of type **(a)** have the same cardinality.

Next consider the function $x \mapsto e^x$, which produces a bijective correspondence between $\mathbb{R}$ and $(0, \infty)$.

Finally consider the function $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, which is also bijective. This demonstrates how all of the open intervals are equivalent. $\square$

4. Base β Expansions

Let A be a set. A *sequence* in A is a function $a : \mathbb{Z}^+ \to A$. We write a_i to mean $a(i)$, and we write $\{a_i\}_{i=1}^{\infty}$, or simply $\{a_i\}$, to denote the function a.

Let β be an integer such that $\beta \geq 2$, and let $\mathbb{N}_\beta = \{0, 1, \ldots, \beta - 1\}$. Let $O = (0, 1)$ be the open unit interval in the real line. We are interested in relating the set of sequences in $\mathbb{N}_\beta$, which is denoted by $\mathcal{F}(\mathbb{Z}^+, \mathbb{N}_\beta)$, to the set O.

Define a function

$$\mu : \mathbb{Z} \to \mathbb{N}_\beta \quad \text{by} \quad \mu(n) = r,$$

where $n = \beta q + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < \beta$.

Define a function

$$\zeta : \mathbb{R} \to \mathbb{Z} \quad \text{by} \quad \zeta(x) = \max\{n \in \mathbb{N} \mid n \leq x\}.$$

For each $k \in \mathbb{Z}^+$, define a function

$$\delta_{\beta,k} : \mathbb{R} \to \mathbb{N}_\beta \quad \text{by} \quad \delta_{\beta,k}(x) = \mu(\zeta(\beta^k x)).$$

This induces a function

$$\delta_\beta : O \to \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_\beta) \quad \text{by} \quad \delta_\beta(x) = \{\delta_{\beta,k}(x)\}_{k=1}^{\infty}.$$

Then δ_β is an injective function, and we call $\delta_\beta(x)$ the *base β expansion* of x.

Construct a partial inverse to δ_β as follows.

Let $\{a_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{N}_\beta$ and set $B = \{\sum_{i=1}^{k} \frac{a_i}{\beta^i} \mid k \in \mathbb{N}\}$. Then $B \subset O$, and in particular, B is a bounded set of real numbers. Set $b = \sup(B)$. For most sequences, $\delta_\beta(b) = \{a_i\}_{i=1}^{\infty}$.

Call a sequence $\{a_i\}_{i=1}^{\infty}$ in $\mathbb{N}_\beta$ a *duplicator* if there exists $N \in \mathbb{N}$ such that $a_i = \beta - 1$ for all $i > N$. These are the only sequences which are not in the image of the function δ_β. If $S = \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_\beta) \setminus \{\text{ duplicators }\}$, then $\delta_\beta : O \to S$ is bijective.

5. Uncountability

Proposition 9. *The set $\mathbb{R}$ of real numbers is an uncountable set.*

Proof. Since $O = (0,1) \subset \mathbb{R}$, it suffices to show that O is uncountable.

Let $\beta = 10$ so that we consider base 10 expansions of the elements in O, and let μ, ζ, and δ_β be as in the previous section.

Let $f : \mathbb{N} \to O$ be any function; we will show that f is not surjective. Set

$$a_i = \begin{cases} 3 & \text{if} \quad \beta_i(f(i)) \neq 3; \\ 6 & \text{if} \quad \beta_i(f(i)) = 3. \end{cases}$$

Set $B = \{\sum_{i=1}^k \frac{a_i}{10^i} \mid k \in \mathbb{N}\}$. Then $B \subset O$, and in particular, B is a bounded set of real numbers. Set $b = \sup(B)$. Then b is not in the image of f. $\square$

We can be even more precise than this.

Proposition 10. $|\mathbb{R}| = 2^{\aleph_0}$.

Proof. Again let $O = (0,1)$. Since $|\mathbb{R}| = |O|$, it suffices to prove that the cardinality of O equals that of $\mathcal{F}(\mathbb{Z}^+, \mathbb{N}_2)$.

First construct a function

$$f : \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_2) \to O \quad \text{by} \quad f(a_i) = \sup\Big\{ \sum_{i=1}^k \frac{a_i}{10^i} \mid k \in \mathbb{Z}^+ \Big\}.$$

This function is injective.

Next consider that $\delta_2 : O \to \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_2)$ is injective.

By the Schoeder-Bernstein theorem, there exists a bijective function $O \to \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_2)$. $\square$

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